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ON THE THEORY OF VISCOELASTICITY OF STRUCTURALLY INHOMOGENEOUS MEDIA*(**)

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The quasi-static problem for the linear theory of viscoelasticity for structurally inhomogeneous media is stated for the case of displacements and stresses. A method of solving such a problem for media with perodic structure is given. "Canonical" viscoelastic operators are introduced for composites in which the reinforcement and binder possess viscoelastic properties, and experiments for determining the kernels corresponding to these operators are described. A method of solving problems for such composities that may be used to determine microscopic displacements and stresses is described.

1. We consider a linear nonhomogeneous viscoelastic medium for which the relation between the stresses σ_{ij} and deformations ε_{ij} has the form /1/

$$\sigma_{ij} = \int_0^t C_{ijkl}(x, t, \tau)\, \varepsilon_{kl}(\tau)\, d\tau \equiv \hat{C}_{ijkl}(x)\, \varepsilon_{kl} \tag{1.1}$$

The tensor of fourth rank $C_{ijkl}(x, t, \tau)$ is called the tensor of the relaxation kernels and in the case of a deformable solid, may be represented in the form

$$C_{ijkl}(x, t, \tau) = \overset{\circ}{C}_{ijkl}(x)\, \delta(t - \tau) - C^1_{ijkl}(x, t, \tau) \tag{1.2}$$

where $\delta(t)$ is a delta function and $C^1_{ijkl}(x, t, \tau)$ a regular component of the relaxation kernel tensor /2/. If conditions (1.2) are satisfied, the determining relations (1.1) may be solved for the deformations thus:

$$\varepsilon_{ij} = \int_0^t J_{ijkl}(x, t, \tau)\, \sigma_{kl}(\tau)\, d\tau \equiv \hat{J}_{ijkl}(x)\, \varepsilon_{kl} \tag{1.3}$$

where an additive component may be extracted from the creep kernel tensor $J_{ijkl}(x, t, \tau)$ in the form of a delta function

$$J_{ijkl}(x, t, \tau) = \overset{\circ}{J}_{ijkl}(x)\, \delta(t - \tau) + J^1_{ijkl}(x, t, \tau) \tag{1.4}$$

Here the regular components of the relaxation and creep kernel tensors are related as follows:

$$\int_\tau^t J^1_{ijkl}(x, t, \tau)\, C^1_{klmn}(x, \xi, \tau)\, d\xi + J_{ijkl}(x)\, C^1_{klmn}(x, t, \tau) = J^1_{ijkl}(x, t, \tau)\, \overset{\circ}{C}_{klmn}(x)$$

$$\int_\tau^t C^1_{ijkl}(x, t, \tau)\, J^1_{klmn}(x, \xi, \tau)\, d\xi + \overset{\circ}{C}_{ijkl}(x)\, J^1_{klmn}(x, t, \tau) = C^1_{ijkl}(x, t, \tau)\, \overset{\circ}{J}_{klmn}(x)$$

The number of independent components of the relaxation and creep kernel tensors depends on the type of anisotropy of in question material /1/. An explicit relation between these kernels and the coordinates x for nonhomogeneous viscoelastic materials may be due to the strucutre of the material, the dependence between its properties and a nonuniform temperature

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field, and also the properties of nonuniform aging (*).

If the material is nonaging, the relaxation and creep kernels will be of different types and equations (1.1) and (1.3) may be written in the form /2/

$$\sigma_{ij}=\int_0^t R_{ijkl}(x,t-\tau)\,d\varepsilon_{kl}(\tau)\equiv R_{ijkl}^{\vee}(x)\,\varepsilon_{kl} \quad (**)$$
$$\varepsilon_{ij}=\int_0^t \Pi_{ijkl}(x,t-\tau)\,d\sigma_{kl}(\tau)\equiv \Pi_{ijkl}^{\vee}(x)\,\sigma_{kl} \tag{1.5}$$
$$C_{ijkl}^{1}(x,t)=-\frac{\partial}{\partial t}R_{ijkl}(x,t),\ R_{ijkl}(x,0)=C_{ijkl}^{\circ}(x)$$
$$J_{ijkl}^{1}(x,t)=\frac{\partial}{\partial t}\Pi_{ijkl}(x,t),\ \Pi_{ijkl}(x,0)=\Pi_{ijkl}^{\circ}(x)$$

If the material is isotropic, the relaxation kernel tensor may be represented in the form /3/

$$C_{ijkl}(x,t,\tau)=[\Gamma_1(x,t,\tau)-{}^1/_3\,\Gamma(x,t,\tau)]\,\delta_{ij}\,\delta_{kl}+\Gamma(x,t,\tau)\,(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$$

where $\Gamma(x,t,\tau)$ is the shear relaxation kernel and $\Gamma_1(x,t,\tau)$ the bulk relaxation kernel. Analogously, we may represent the tensors $J_{ijkl}(x,t,\tau)$, $R_{ijkl}(x,t)$, and $\Pi_{ijkl}(x,t)$.

If the relaxation and creep kernel tensors are piecewise-braking functions of the coordinates x, the material is called a linear viscoelastic composite. A two-component composite in which one of the components (the reinforcement) is an isotropic, elastic material and the other component (the binder) is an isotropic voscoelastic material with nonrelaxing volume /1/ is called a simple viscoelastic composite.

For a non-aging binder, the relation between the stresses and deformations has the form

$$s_{ij}=3K_c\omega^{\vee}e_{ij},\quad e_{ij}=\frac{1}{3K_c}\pi^{\vee}s_{ij},\quad \sigma=K_c\theta$$
$$e_{ij}\equiv\varepsilon_{ij}-{}^1/_3\theta\delta_{ij},\ s_{ij}\equiv\sigma_{ij}-\sigma\delta_{ij},\ \theta\equiv\varepsilon_{kk},\ \sigma\equiv{}^1/_3\sigma_{kk}$$

Here K_c is the modulus of compression of the binder and $\omega^{\vee}$ and $\pi^{\vee}$ are two reciprocal integral operators (1.5) that characterise the kernels $\omega(t)$ and $\pi(t)$, respectively. Consequently, the mechanical properties of a non-aging simple viscoelastic composite is described by the relaxation kernel $\omega(t)$ and elastic constants K_c and E_a, which is Young's modulus of the reinforcement, and ν_a, which is Poisson's coefficient of the reinforcement. If $\omega(t)=\omega=\mathrm{const}$, the composite is elastic.

2. Let us consider the quasi-static problem for a nonhomogeneous viscoelastic medium (problem A) which consists in solving three equations for the displacement vector u (we assume that there are no body forces) under the assumption that boundary conditions are satisfied, for example, a mixed type of boundary conditions in which displacements u° are specified on the part Σ_1 of the boundary of the body bounding the volume V_0, and loads S° are specified on the rest of the boundary Σ_2:

$$[C_{ijkl}^{\wedge}(x)\,u_{k,l}]_{,j}=0 \tag{2.1}$$
$$u_i|_{\Sigma_1}=u_i^{\circ};\quad C_{ijkl}^{\wedge}(x)\,u_{k,l}n_j|_{\Sigma_2}=S_i^{\circ} \tag{2.2}$$

If there is no regular part in the relaxation kernels (1.2), the medium will be elastic. We will refer to the problem (2.1), (2.2) for an elastic medium as problem A_y. Suppose we have found a solution of problem A_y under the condition that displacements are specified on the entire boundary Σ, i.e., in (2.2) $\Sigma_1=\Sigma,\ \Sigma_2=0$, further u_i° are linear functions of the coordinates. We let $\varepsilon_{ij}^{\circ}(x)$ denote the deformation of the resulting solution, and $\sigma_{ij}^{\circ}(x)$ the stresses, then average them with respect to the volume, obtaining $\langle\varepsilon_{ij}^{\circ}\rangle$, $\langle\sigma_{ij}^{\circ}\rangle$, where, for example,

$$\langle\varepsilon_{ij}^{\circ}\rangle\equiv\frac{1}{V}\int_V\varepsilon_{ij}^{\circ}(x)\,dV,\quad\langle\sigma_{ij}^{\circ}\rangle=\mathring{h}_{ijkl}\,\langle\mathring{\varepsilon}_{kl}\rangle \tag{2.3}$$

The quantities h_{ijkl}° form an effective elasticity modulus tensor. Solving problem A with the above boundary conditions, we similarly determine the effective relaxation kernel

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*) N.Kh. Arutiunian, Theory of creep of nonhomogeneously aging bodies. Offprint of the Institute of Problems of Mechanics, Akad. Nauk USSR, Moscow, No.170, 1981.

**) In Russian original the letter σ is omitted-Ed.

tensor $h_{ijkl}(t, \tau)$.

If only the loads

$$\sigma_{ij}n_j|_\Sigma = S_i^\circ \qquad (2.4)$$

are given on the boundary, the problem is sometimes more convenient to formulate in terms of stresses. For this purpose, we consider at tensor of third rank symmetric in its first two subscripts:

$$\begin{aligned} E_{ijk} \equiv \varepsilon_{ij,k} + \delta_{kl}(^1/_2\theta_{,j} - \varepsilon_{jl,l}) + \delta_{kj}(^1/_2\theta_{,i} - \varepsilon_{il,l}) + \\ \xi_{ij}(\varepsilon_{kl,l} - \theta_{,k}) + M_i(q)\delta_{jk} + M_j(q)\delta_{ik} - \xi_{ij}M_k(q) \end{aligned} \qquad (2.5)$$

where $M(q)$ is some arbitrary linear vector, that is, an operator over the vector q that vanishes on the boundary

$$q_i|_\Sigma = 0; \; q_i \equiv \sigma_{ik,k} \qquad (2.6)$$

The quasi-static problem for a nonhomogeneous viscoelastic medium (problem B) consists in solving the six equations

$$E_{ijk,k}(\sigma) = 0 \qquad (2.7)$$

for the six independent components of the tensor σ_{ij} (in which the tensor E_{ijk} of (2.5) is expressed in terms of stresses by means of formula (1.3)) under the six boundary conditions (2.4) and (2.6) /1/. If there is no regular part in the creep kernels (1.4), the medium will be elastic. We will refer to the problem (2.7), (2.4), (2.6) for an elastic medium as problem B_y. If its solution is averaged over the volume in accordance with the first formula of (2.3) (assuming the loads S° (2.4) are constant), we may then write

$$\langle \varepsilon_{ij}^\circ \rangle = \overset{\circ}{H}_{ijkl} \langle \sigma_{kl}^\circ \rangle$$

where the quantities $\overset{\circ}{H}_{ijkl}$ form what is known as the effective compliance tensor. A definition of the effective creep kernel tensor $H_{ijkl}(t, \tau)$ may be given in an obvious way.

We may prove that the tensors $\overset{\circ}{h}_{ijkl}$ and $\overset{\circ}{H}_{ijkl}$ (that is, $h_{ijkl}(t, \tau)$ as well) and $H_{ijkl}(t, \tau)$ are reciprocal. Expressions for these tensors have been found for some media /4 – 7/.

3. The theory based on replacing the elasticity modulus tensor $\overset{\circ}{C}_{ijkl}(x)$ (the relaxation kernel tensor $C_{ijkl}(x, t, \tau)$) in problem A_y (or A) by the effective quantities $\overset{\circ}{h}_{ijkl}$ ($h_{ijkl}(t, \tau)$) is called effective modulus theory. In this theory, the problem (2.1), (2.2) for a nonhomogeneous viscoelastic medium (isotropic or anisotropic) is replaced by the problem for a homogeneous anisotropic viscoelastic medium:

$$\hat{h}_{ijkl}v_{k,lj} = 0 \qquad (3.1)$$

$$v_i|_{\Sigma_1} = u_i^\circ, \quad \hat{h}_{ijkl}v_{k,l}n_j|_{\Sigma_2} = S_i^\circ \qquad (3.2)$$

where v is some mean displacement field /4,5,8,9/.

In many cases, for example when estimating the strength of a composite material, it is important to know not only the average diffuse displacement and stress field, but also the stresses within each component forming the composite (what are known as microscopic stresses).

To determine the microscopic deformations and microscopic stresses in composites with a periodic structure, a method (/1/, p.269) based on the notion of an average of differential equations with periodic coefficients /10/ is highly recommended. By means of this method, certain boundary-value problems of the theory of elasticity may be solved exactly /11/. Interestingly, even the first approximation, i.e., the approximation based on the effective modulus theory with small corrections, in many cases describes in a sufficiently complete manner the nature of the deformation of the nonhomogeneous medium.

In this case, the solution of problem A, i.e., problem (2.1), (2.2), is found approximately in the form

$$u_i(x, \xi) = v_i(x) + \alpha\hat{N}_{ijk}(\xi)v_{j,k}(x), \quad \xi \equiv x/\alpha \qquad (3.3)$$

where α is a small parameter equal to the ratio of the period of the structure and the characteristic dimension of the entire nonhomogeneous body. The quantities $N_{ijk}(\xi, t, \tau)$ (local relaxation kernels) are periodic functions of the fast coefficients ξ and are determined by solving the system of differential equations

$$\nabla_j[\hat{C}_{ijkl}(\xi)\nabla_l N_{mnk}] = -\nabla_j C_{ijmn}(\xi, t, \tau) \qquad (3.4)$$

where ∇_j represents the symbol of derivative with respect to the coordinates ξ. To uniquely determine the functions $N_{ijk}(\xi, t, \tau)$, we must set

$$\langle N_{ijk}(\xi, t, \tau)\rangle = 0 \qquad (3.5)$$

The effective relaxation kernel tensor is found using the formulas /1/

$$h_{ijmn}(t,\tau) = \langle C_{ijkl}^{\frown}(\xi)\nabla_l N_{mnk} + C_{ijmn}(\xi,t,\tau)\rangle \tag{3.6}$$

Note that the effective relaxation kernel tensor must also be found in the effective modulus theory (3.1), (3.2). In this case, there is a method of finding these kernels (eqn.(3.6)), whereby we first determine the local relaxation kernels (eqns. (3.4), (3.5)) and in (3.3) add the second term which takes into account the microscopic stresses to the first term, which is the solution in the effective modulus theory. Note, too, that the boundary conditions in this approximation are satisfied approximately (cf. (3.2)), as in the effective modulus theory. In the case of a rectilinear boundary, there exist techniques for exactly satisfying the boundary conditions /11, 12/. The effective relaxation and creep kernels as well as the local relaxation kernels for a laminar composite have been described explicitly /1/. For composites with more complex strucute, these quantities may be determined only approximately, and it is best to use empirical analytic expressions(*).

4. There are no general-purpose effective solution methods for the problem (3.1), (3.2) of the theory of viscoelasticity for an anisotropic medium. If the composite is a simple one, the quantities $h_{ijkl}(t,\tau)$ depend only on the single operator $\omega^{\wedge}$, so that the numerical realization method for the elastic solution /1/ may be successfully applied; this method is a generalization of the approximation method developed by A.A. Il'iushin /2/. However, composites in which viscoelastic properties are exhibited not only by the binder, but also by the reinforcement, have recently come into widespread use in industry. For such materials, the above method is no longer applicable, since a rational function of two arguments (viscoelastic operators) may not be expanded into simple fractions. Below we describe a technique that may be used to solve the problem (3.1), (3.2) for two-component composites. Suppose K_1 and K_2 are the compression moduli of the first and second component, $\omega_1(t)$ and $\omega_2(t)$ the shear relaxation kernels of the first and second component, while $\pi_1(t)$ and $\pi_2(t)$ the creep kernels corresponding to them.

We let

$$g_\beta^{\smile} \equiv \frac{1}{1+\beta\omega_1^{\smile}}, \quad \psi_\beta^{\smile} \equiv \frac{1}{1+\beta\omega_2^{\smile}} \tag{4.1}$$

and introduce certain 'canonical' operators

$$\begin{aligned} &B^{\smile}(\beta,a) \equiv g_\beta^{\smile} + a\psi_\beta^{\smile} \\ &B_\pi^{\smile}(a) \equiv \pi_1^{\smile} + a\pi_2^{\smile} = \lim_{\beta\to\infty} \beta B^{\smile}(\beta,a) \\ &A_\omega^{\smile}(a) \equiv \omega_1^{\smile} + a\omega_2^{\smile} \end{aligned} \tag{4.2}$$

where a and β are certain numbers. Let us also consider the operators $A^{\smile}(\beta,a)$, $A_\pi^{\smile}(a)$ and $B_\omega^{\smile}(a)$, which are reciprocals of the corresponding operators (4.2).

Then for the case of laminar composites we may write out an expression for the relaxation operators in terms of the canonical operators:

$$\begin{aligned} &h_{1111}^{\smile} = h_{2222}^{\smile} = {}^9/_4 A^{\smile}(2,\alpha/\varkappa)[B^{\smile}(2,\alpha) - {}^1/_3\gamma^{-1}]^2 - \\ &\quad {}^9/_4 B^{\smile}(2,\alpha\varkappa) + {}^3/_2 A_\omega^{\smile}(\alpha\varkappa) + {}^9/_4(1+\alpha\varkappa) \\ &h_{3333}^{\smile} = \gamma^{-2}B^{\smile}(2,\alpha/\varkappa), \quad h_{1122}^{\smile} = h_{1111}^{\smile} - 3A_\omega^{\smile}(\alpha\varkappa) \\ &h_{1133}^{\smile} = h_{2233}^{\smile} = {}^3/_2\gamma^{-1}A^{\smile}(2,\alpha/\omega)[B^{\smile}(2,\alpha) - {}^1/_3\gamma^{-1}] \\ &h_{1212}^{\smile} = {}^3/_2 A_\omega^{\smile}(\alpha\varkappa), \quad h_{1313}^{\smile} = h_{2323}^{\smile} = {}^3/_2\gamma^{-2}A_\pi^{\smile}(\alpha/\varkappa) \\ &\varkappa \equiv K_2/K_1, \ \alpha \equiv (1-\gamma)/\gamma \end{aligned}$$

The remaining components of the relaxation tensor operators (relative to $K_1\gamma$) are equal to zero. Here γ is the thickness of the layer of the first component, relative to the thickness of the entire layer (periodicity cells).

We may analogously write out expressions for the creep operators.

Experiments have been described /2/ for determining the kernels $g_\beta(t)$ and $\psi_\beta(t)$ corresponding to the operators of (4.1).

Let us now describe experiments that may be used to determine the kernels corresponding to the operators $A^{\smile}(\beta,a)$ and $A_\pi^{\smile}(a)$.

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*) V.V. Doroginin, On a solution of spatial static problems of elastic composites, Dissertation Read to a Conference of the Senior Scientific Candidates in Mathematical Physics, Moscow, MGU, 1980, 112 pp.; M.G. Gadzhiev, Effective elasticity modulus tensor of composite material, Moscow, 1979, Dep. VINITI, 20.03, No.968-79; cf. also /13/.

Let a spring depicted as in Fig.1 has rigidity k. Samples of the first and second components of the composite with ratio between the length and area of the cross-section of f_1 and f_2, respectively, are successively applied to this spring.

Suppose that the condition

$$\frac{1}{k} = \frac{f_1}{9K_1} + \frac{f_2}{9K_2}$$

is observed.

Then the relation between the force Ω stretching the samples and the displacements u will have the form

$$Q = \frac{1}{b_1\pi_1^{\smile} + b_2\pi_2^{\smile}}\, u \left(b_\alpha = \frac{2f_\alpha}{9K_\alpha},\ \alpha = 1,2\right)$$

from which we may at once determine the operator $A_\pi^{\smile}(a)$.

If we consider a more complex system consisting of a parallel connection of a spring of rigidity k and system of samples formed by both components connected successively to springs of rigidities k_1 and k_2 (Fig.2), then, if the conditions

$$k = 1/c_1 + 1/c_2$$
$$c_\alpha \equiv 1/k_\alpha + f_\alpha/(9K_\alpha),\ \alpha = 1,\ 2;\ c_1/b_1 = c_2/b_2 \equiv \beta$$

are observed, where the quantities b_α are determined as above, the relation between the displacement u and force Q will have the form

$$u = \frac{c_1}{g_\beta^{\smile} + \frac{c_1}{c_2}\psi_\beta^{\smile}}\, Q$$

from which the kernel corresponding to the operator $A^{\smile}(\beta, a)$ may be determined.

We proceed as follows if we wish to determine an approximate solution of problem (3.1), (3.2). If we are able to solve the corresponding elastic problem for an anisotropic medium analytically, we will obtain in the solution an expression of the type $\varphi(\cdot)S$, where S is a known quantity, while $\varphi(\cdot)$ denotes a function of the elastic anisotropic moduli. Substituting for these moduli their expressions in terms of the quantities K_1, K_2, ω_1, ω_2 and γ, we approximate $\varphi(\omega_1, \omega_2)$ by some analytic expression of the canonical operators. The coefficients of this analytic approximation may be found using, for example, the method of least squares /1/.

If the analytic solution of the corresponding elastic problem is not known, it may be found numerically or experimentally, using numerical realizations of the elastic solution /1/. For this purpose, we assume that one of the components of the displacement vector (denote it by v) has the form

$$v = A_1 + A_2\omega_1 + A_3\pi_1 + A_4\omega_2 + A_5\pi_2 + A_6 g_\beta + A_7\psi_\beta + \frac{A_8}{g_2 + a\psi_2} + \frac{A_9}{\pi_1 + b\pi_2} \tag{4.3}$$

and solve the problem of the theory of elasticity numerically for different Poisson's ratios of the material of the reinforcement and binder. Then, at each point of the network we solve an algebraic system of equations for the unknowns occurring in (4.3): $A_1, \ldots, A_9$, β, a, and b. Next, the solutions of the problem (3.1), (3.2) is expressed in quadratures with respect to time /1/.

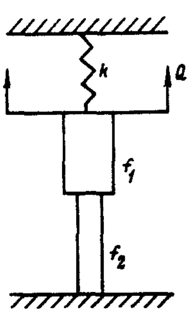

Fig.1

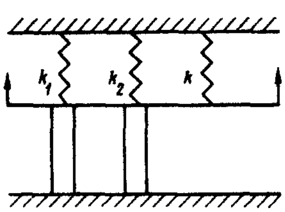

Fig.2

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